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# Human capital as a risky asset and the effect of uncertainty on the decision to invest

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## Abstract

In this paper, we analyse the human capital accumulation process over the life cycle of individuals under uncertainty. To do so, we develop a continuous time dynamic programming model which takes into account several sources of uncertainty, concerning the human capital accumulation process and the labour market. We also introduce some macro exogenous variables to take into account intertemporality of decisions. We first determine the structure of human capital investment in a general setting. Then, we specify individual preferences to obtain explicit solutions, and we produce an in-depth study of each source's effect of uncertainty on human capital investment. As a special case of state variable, we explicitly take into account unemployment risk. We show that the global effect of uncertainty is negative, except if a sufficiently high risk premium exists.

*key words* : human capital, life cycle, risk, stochastic optimal control.

*JEL Classification* : I20, J24, C61, D99, G11.

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# 1 Introduction

Since the pioneering studies of Schultz (1961) and Becker (1962) in the area of education, schooling and training have been considered as investments in human capital in the same way as physical capital. But for many years, the risky nature of this investment was overlooked in the analysis. Over the past two decades, however, European labour markets have seen the rise of an often chronic and sometimes long-term form of youth unemployment, accompanied by wage downgrading at the time of first hiring. The vulnerability of the young people and the uncertainty they face at the beginning of their working life are expressed not only by the increased difficulties in finding a job which corresponds to their level of schooling but also by employment rates which are much more sensitive to the economic situation than those found among other categories of the labour force. In the area of education, we also observe a diversification of training streams and pathways. At aggregate level, this increased heterogeneity may accentuate the uncertainty about the quality of the pathways. At individual level, it tends to blur employers' perceptions of the students' productivity potentials.

As a result, the predictions of models based on the hypothesis of perfect foresight about the future value of the human capital accumulated seem less and less adapted to the European observed data. It thus seems necessary to resituate the problem of human capital investment, as well as that of the efficiency of the educational systems stemming from it, in a more general framework which takes uncertainty into account. The question which then arises is to what extent and in what forms uncertainty influences decisions to invest in human capital.

The first rigorous theoretical analysis of this question was proposed by Levhari and Weiss (1974) in a two-period model. The uncertainty about future wages, or in other words, the return to the human capital investment, is assumed to come from two sources. The first has to do with the learning process in the educational system and covers a group of exogenous individual and collective features such as, for example, the students' scholastic aptitude, but also the quality of the courses, the schools, the teachers, the scholar paths and so on. The second arises from the labour market and covers the future conditions of labour supply and demand. These two type of sources of uncertainty, which are intuitive and fairly realistic, are not present in the model specifications. Indeed, the sources of uncertainty are not separated out; rather, they are aggregated and represented by a single random variable. Uncertainty is defined by the variance of earnings, in the sense of Rothschild-Stiglitz (1970). The effect of uncertainty is studied on the basis of the correlation between the average and future marginal returns to human capital. If this correlation is positive, or if the variance of the gains increases with the level of schooling (which amounts to the same thing), the anticipated return to human capital will be greater than that to a financial asset presumed to be certain. In this case, risk averse individuals will protect themselves by reducing their human capital investment. A negative correlation between average and marginal returns has the opposite effect: an increase in the level of schooling will reduce the variance of the future gains. In this case, investment in human capital is encouraged when risk increases. The hypothesis generally employed in the empirical literature, however, is that of an increasing risk<sup>1</sup> (increasing variance of earnings), which leads to the assumption

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<sup>1</sup>See Low and Ormiston (1991) for empirical evidence on the NLS data.

of a lower education level in a situation of uncertainty.

More recently, Kodde (1986) has provided empirical results which, contradicting Levhari and Weiss's prediction, indicate on the contrary the existence of a positive relationship between uncertainty and investment in human capital. In an attempt to take this empirical observation into account, Snow and Warren (1990) developed an extension of the Levhari and Weiss model by including the hypothesis of an endogenous labour supply. This permits the introduction of an income effect which may make the relationship between risk and investment ambiguous. In other words, the investment may increase or decrease with risk.

In our view, this ambiguity of the effect of risk on investment stems from the aggregation of the different sources of uncertainty through a single random parameter capturing all the effects. It might be thought a priori that these different sources of uncertainty have an unequal, if not contradictory, effect on the investment in human capital. In the models already cited, moreover, the fact that the individual's planning horizon is reduced to two periods does not allow the intertemporal nature of the human capital investment to be brought out.

In order to get beyond these two limitations, we propose a dynamic programming model in continuous time which allows us not only to study the individual's behaviour over the whole of his or her life cycle but also to break down and separately identify the effects of the different sources of uncertainty.

In the next section, we present the general principles of the problem of human capital investment in a stochastic context. Section 3 is then devoted to the specification of our model. In section 4, we present the results of the model in the general case where individual preferences are not specified. In section 5, after identifying the individual preferences, we discuss the effect of the different sources of uncertainty on optimal investment in human capital. Our conclusions are presented in section 6.

## **2 The continuous-time stochastic model of human capital: general setting**

The study of the dynamic nature of human capital investment in a context of uncertainty can, as in the case of certainty, take two directions. In the first (e.g., Fan [1993], Hogan and Walker [2002]), the theory of real options is applied to the problem of education, with a model of schooling choice transposed to the uncertain case. In the second (Williams [1979]), the portfolio theory is applied to the problem of investment choice over the individual's entire life cycle.

The first group of studies develop a pure schooling model in which the individual has to decide on the optimal, definitive date for stopping his or her studies. Here, education is considered to be an irreversible investment<sup>2</sup>. The problem facing the individuals is the following: as long as they remain in the educational system, they have at each date the option to leave school and enter the labour market at a wage rate which depends stochastically on the amount of

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<sup>2</sup>The stochastic versions of the optimal stopping problem are analysed in Kamien and Schwartz (1991). The application of these techniques to physical capital investment is reviewed in Dixit and Pindyck (1994).

time spent in school. Once that option is exercised, they can no longer return to the educational system and will, throughout their life cycle, receive an income which depends uniquely on the accumulated schooling. In the presence of risk, individuals are encouraged to postpone their exit from the school system because of the irreversibility of their choice. Indeed, by remaining in school, they have the option of leaving during the following period with the aim of taking advantage of a ‘better draw’ in the distribution of returns. They also have the choice of remaining in school in order to avoid a ‘bad draw’. For this reason, uncertainty increases the potential advantage of the option. But if, in anticipation of a low wage, individuals do not exercise their option, the loss of value of the option remains unchanged. This asymmetry of the effects of uncertainty on the value of the option incites individuals to postpone their exit from the school system even longer when uncertainty about wages is great<sup>3</sup>.

Thus, the duration of schooling is an increasing function not only of the anticipated return but also of the risk associated with education. This model introduces micro-economic elements in the analysis of the phenomenon of continued studies, by emphasising the protective role of education in face of the risks existing at the time of labour-market entry. Nonetheless, the findings of this kind of model must be qualified in the light of the restrictive hypotheses on which it is based. First of all, the Hogan and Walker (2002) model, for example, may be seen as a very specific case of the human capital model, namely a *pure schooling* model in which labour supply is presumed to be exogenous. This means that the duration of schooling is the only variable which the individual controls. As a result, the model rules out any possibility of training associated with employment. During the employment period, wages are presumed to increase at an exogenous rate which is identical for all workers. Thus, the average wage differentials observed over the individuals’ life cycles are explained solely on the basis of schooling levels. Any possibility of changing the initial situation through post-schooling investment in human capital is excluded. So, the question of investment over the life cycle becomes irrelevant because the individual’s future situation is definitively established at the time of the exit from the school system.

Last of all, the model does not distinguish the sources of uncertainty. Since the process of human capital accumulation is not specified, all uncertainty is presumed to come from the labour market through wages. The risky content of school-based learning is totally absent.

Here, we would point out that the diversification of training pathways increasingly encountered in Western economies can also affect uncertainty about the students’ productive capacities. This observation suffices to justify taking into account the risk which may exist within the education process. This second path of analysis corresponds to the approach taken by Williams (1979) in a different but more general theoretical context. Williams proposes a portfolio model much closer to the initial human capital model. In fact, he generalises the basic model, where the individual makes a trade-off between non-risky activities, to the case of a trade-off between risky activities. The object of the individual choice is not so much that of the optimal duration of studies as that of the optimal intensity of investment in human capital over the life cycle. This comes from the fact that, for Williams, human capital cannot be reduced solely

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<sup>3</sup>The proof of this is provided in the appendix of Hogan and Walker (2002).

to initial training. Post-schooling training, especially on-the-job training, is another form of human capital investment taken into account. Here, pure schooling appears to be a special case of a more general process of human capital accumulation. Williams's frame of analysis appears also more general, to the extent that, on the one hand, the labour supply is endogenous and, on the other, different sources of uncertainty, involving both the educational process and the wage, are taken into account. More precisely, four sources of uncertainty are distinguished. A first source bears on the value of the financial assets in which the individual can invest on the financial market. Two others stem from the process of human capital accumulation: one concerns the efficiency of school-based learning and the other, the depreciation rate of the human capital. The last source of uncertainty has to do with the wage rate by level of skills. The co-variances between the different risky variables play a central role in the analysis, notably in the determination of the overall effect of risk on the investment in human capital. Contrary to Hogan and Walker (2002), Williams concludes that risk has a negative effect: individuals are led to reduce their human capital investment when the risk associated with education increases.

In the section which follows, we show that this result stems from the (ad hoc) hypothesis of the independence of the different markets. By forcing the nullity of certain covariances, particularly those linking the risks relative to the knowledge-acquisition processes to the risk over wages, Williams implicitly brings all the weight of uncertainty to bear on the learning process. This hypothesis is difficult to defend to the extent that the very essence of the economics of education (Willis and Rosen 1979) lies in the study of the mechanisms articulating the schooling process and its recognition on the labour market. The observable and non-observable individual features explaining the differences in education levels also account in part for the differences in wages (Willis 1986, Griliches 1977).

On the basis of all of these criticisms, we propose a theoretical framework which generalises Williams' (1979) model of human capital as a risky asset, in several directions. We first deduce an optimal structure of human capital investment from completely general diffusion processes. We also remove hypothesis of independence between random variables, especially those describing the process of acquiring knowledge and those relating to the labour market. Finally, one of the main contribution of our model is the central place we give to intertemporal dimension of stochastic processes and its consequences on individual behavior. Indeed, for a long time, intertemporal dimension played a minor part in finance, because everything was supposed to be stationary. But an extensive literature has been developed to show that drift and volatility of stochastic processes are changing over time, and to some extent predictable because affected by state variables. Now, we know that intertemporal dimension yields a welfare gain or a welfare cost, which need to be integrated into the decision making. But Williams (1979) model supposes geometric brownian motions. Constant parameters of processes do not allow intertemporal hedging components into choices under uncertainty<sup>4</sup>. In this paper, we are interested in intertemporal hedging, because it corresponds to the reality observed in the arena of education. A lot of people continue some studies that are not immediately valued on the labour market, but in fine they quickly find a stable job. High qualified people may

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<sup>4</sup>During time, stochastic processes remain at their mean value. So there is nothing to hedge.

accept to begin with low paid job to get a better access to stable labour market afterwards. Or, on the contrary, they may prefer a period of unemployment to many bad jobs, to later access to a job which corresponds to their level of education. Whatever the strategy against difficulties on the labour market, it seems that indirectly education choices may partly control some variability factors of future human capital return. So far, to the best of our knowledge, neither Williams (1979) nor everybody else have attempted to report such evidence in a human capital model. Our model, full of these relations, provides an optimal structure of human capital that allows a better understanding of individual behavior in front of risk.

### 3 Specification of the model

In this section, we present a dynamic stochastic model of human capital accumulation over the life cycle. The uncertainty here stems from four sources as in Williams's (1979) model. Uncertainty intervenes in the constraints of the accumulation of human capital and financial wealth, which is why the first step in this study consists of correctly deriving the stochastic versions of these accumulation equations.

A common feature of all human capital models is that an individual may improve his future situation on the labour market, by spending time in education activities. Today investment in human capital is supposed to increase productive abilities of individuals, which will be paid later on the labour market. However, such investment is costly, since during time allocated to education, individuals give up the wage they would receive by working immediately. This trade-off between immediate certain wage and future higher wage, but uncertain, is in the center of the human capital model.

In the model, the proportion of time which he or she devotes to training during the period  $t$  is measured by  $\lambda(t)$ , while  $l(t)$  designates the proportion of time allocated to leisure activities during the period  $t$ . Thus,  $(1 - \lambda(t) - l(t))$  is the proportion of remaining time devoted to work.

The future human capital stock<sup>5</sup>  $K(t + \Delta t)$  is equal to the current stock  $K(t)$  plus the new human capital produced during the period  $t$ :  $\theta(t, t + \Delta t)\lambda(t)k(t)$ . This production of human capital is supposed to depend linearly of the proportion of time devoted to education  $\lambda(t)$  and a parameter  $\theta(t, t + \Delta t)$  which measures the efficiency of training. It is also necessary to subtract the depreciation of the human capital stock during the period  $t$ , which is equal to  $\delta(t, t + \Delta t)k(t)$ :

$$K(t + \Delta t) = K(t) + \theta(t, t + \Delta t)e(t)K(t) - \delta(t, t + \Delta t)K(t) \quad (1)$$

By multiplying each of the two members of the equation above by " $\omega$ ", the market price of the human capital, one obtains the money value of the stock of human capital on the labour market :

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<sup>5</sup>for the period  $t + \Delta t$ .

$$\omega(t + \Delta t) K(t + \Delta t) = \omega(t + \Delta t) [1 + \theta(t, t + \Delta t) \lambda(t) - \delta(t, t + \Delta t)] K(t) \quad (2)$$

Assuming that  $k(t + \Delta t) = \omega(t + \Delta t) K(t + \Delta t)$ , the value of the stock of future human capital actually used in the future occupation can be rewritten in the following form:

$$k(t + \Delta t) = \frac{\omega(t + \Delta t)}{\omega(t)} [1 - \delta(t, t + \Delta t) + \theta(t, t + \Delta t) \lambda(t)] k(t) \quad (3)$$

Written in this form, this equation indicates that the value of the stock of human capital of the period  $t + \Delta t$ , depends not only on the variation of the volume of human capital, expressed by the term within the square brackets, but it depends also on the variations, between the dates  $t$  and  $t + \Delta t$ , of the value  $\omega$  of human capital stock on the labour market, expressed at the same time by. In this model,  $k(t)$  and  $k(t + \Delta t)$  may thus be defined respectively as the maximum current and future income which the individual can expect on the labour market. To display the stakes of education, in the human capital model, it is important to remind that individuals do not allocate all their time to work. As we have already seen above individuals also spend some time to leisure and education. Thus, current effective labour income is the following :

$$y(t) = (1 - \lambda(t) - l(t))k(t) \quad (4)$$

Combination of equation (3) and (4) shows the trade-off between education and work : during education time, the individual gives up a part of the labour income :  $\lambda(t)k(t)$ , he would have received by working immediately (equation 4). On the other hand, current education increases future income (equation 3) in a proportion  $\frac{\omega(t + \Delta t)}{\omega(t)} \theta(t, t + \Delta t) \lambda(t) k(t)$ , which is uncertain. In the model, uncertainty bears indeed on the parameters  $\theta(t, t + \Delta t)$  and  $\delta(t, t + \Delta t)$ , relating to the process of accumulation of human capital in volume, as well as on the parameters  $\omega(t + \Delta t)$ , relating to the future value of the human capital on the labour market.

The stochastic parameter  $\theta(t, t + \Delta t)$  represents the uncertainty about the gross efficiency of human capital investment. It largely depends on the individual's real cognitive ability (at school and on the job). It also takes into account all other unobservable inputs entering the education process, such as quality of schools, teachers... and on the job training. The random parameter  $\delta(t, t + \Delta t)$  represents the unknown rate of human capital depreciation in each period.

The random parameter  $\omega(t + \Delta t)$  corresponds to the future wage rate of the human capital accumulated, unknown at the time  $t$ . We suppose that the individual perfectly observes the current wage  $\omega(t)$  corresponding to his level of human capital, but doesn't know the distribution of future wages. The randomness of the distribution of wage rates is a manner of characterizing future trends in the labour market - in particular, the institutional conditions of wage determination - largely uncontrolled by the individual at the time he makes his



decisions. In other words, two individuals having the same characteristics, with the same level of human capital, can perceive different wages in the future, only because they will have obtained a different "draw" in the distribution of wages.

In a world of perfect information, these parameters would be constant, but under uncertainty it is not the case. When we introduce the temporal dimension, the variations of each of the random parameters are presumed, according to Ito's lemma<sup>6</sup>, to follow a diffusion process characterised by the following stochastic differentials :

$$\Delta\theta(t) = \mu_\theta\Delta t + \sigma_\theta\Delta Z(t) \quad (5)$$

$$\Delta\delta(t) = \mu_\delta\Delta t + \sigma_\delta\Delta Z(t) \quad (6)$$

$$\frac{\Delta\omega(t)}{\omega(t)} = \mu_\omega\Delta t + \sigma_\omega\Delta Z(t) \quad (7)$$

$\mu_\theta, \mu_\delta$  and  $\mu_\omega$  represent the instantaneous means of the respective stochastic processes.  $\sigma_\theta, \sigma_\delta, \sigma_\omega$  and  $\sigma_h$  are  $N + 4$  dimensional vectors<sup>7</sup> which correspond to the instantaneous standard deviations of each stochastic process. They are indicators of risk in Rothschild-Stiglitz' sense<sup>8</sup>.  $Z(t)$  is the standard Wiener process. By definition, it is a random vector of zero mean, zero covariances and variance  $\Delta t$ . In this model, it is also of dimension  $N + 4$ . Notice that, state in this form,  $\frac{\Delta\omega(t)}{\omega(t)}$  measure the labour market rate of return of one unit of human capital invested.

On the basis of these four expressions, by making  $\Delta t$  tend towards 0, by replacing these values in (3) and applying Ito's lemma, we obtain the stochastic expression of the human capital accumulation equation in continuous time<sup>9</sup> :

$$\begin{aligned} \frac{dk(t)}{k(t)} &= [\mu_\omega + (\mu_\theta + \sigma_{\theta\omega})\lambda(t) - \mu_\delta - \sigma_{\delta\omega}] dt \\ &\quad + [\sigma_\omega + \sigma_\theta\lambda(t) - \sigma_\delta]' dZ(t) \end{aligned} \quad (8)$$

This equation establishes that the global return on human capital  $\frac{dk(t)}{k(t)}$  follows a diffusion process which combines linearly the volume and monetary components of the human capital accumulated. The rate of return on human capital also depends linearly on the time devoted to education  $\lambda(t)$ . More precisely, the current allocation to education  $\lambda(t)$  simultaneously increases the mean and the variance of the instantaneous human capital growth rate. Here are described two very important features of the time allocated to education in our model.

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<sup>6</sup>Ito's lemma, also known as the fundamental theorem of stochastic calculus, is the most commonly used result in continuous-time models. It permits the determination of the parameters of any Ito process when the latter depends on a process of the same nature with known parameters. The unidimensional and multidimensional versions of Ito's lemma are given by Rogers (1991), pp. 234-236.

<sup>7</sup>As we see below,  $N + 4$  is the number of stochastic processes in the model.

<sup>8</sup>"a mean-preserving spread".

<sup>9</sup>The calculations used to obtain this equation are presented in their totality in appendix A.1.

On one hand, and in a classical way, investment in education increases expected future earnings over the life cycle. On the other hand, current education also increases the variance of the future rate of return of human capital. That's why education is considered as a risky asset. When the individual invests in human capital, he sacrifices an immediate certain income in order to increase it in the future. But at the same time individuals are running a risk. This feature of the model has important implications to understand the optimal structure of education demand, and subsequent strategies against risks.

On the basis of this equation, we may note that the extent of this educational effect crucially depends on the net productivity parameter  $\theta$ . Only uncertainty about the depreciation of human capital has no weight on the education effect. Unlike Williams (1979), uncertainty about the labour market directly affects the return to the human capital investment through the instantaneous covariance  $\sigma_{\theta\omega}$ . In Williams's model indeed, the adjustments occurring on the labour market and in the educational system are presumed to be independent, which implies the nullity of  $\sigma_{\theta\omega}$  and other covariances appearing in equation (8).

The originality of our work is precisely to reject this assumption of independence, and suppose, on the contrary, the existence of a nonnull relation between the variables associated with the process of human capital accumulation and those associated with the value of human capital on the labour market.  $\sigma_{\theta\omega}$  identifies the relation between the efficiency of training and the level of the wage rates. We suppose that these instantaneous covariances are positive : the most effective individuals, or individuals who have followed the most effective formations, are those who reach higher levels of wages corresponding to their level of human capital. This assumption, rather intuitive, is not contradicted by empirical work. Besides, the aim of a vast econometric literature around the famous "ability bias" is to identify this type of relations

At the same time, individuals are assumed to divide their current financial wealth  $W(t)$  among three elements: consumption expenditures  $c(t)$ , an investment in a riskless asset and an investment in  $N$  risky assets on the financial market. If  $X(t)$  is the proportion of the wealth invested in risky assets,  $y(t)$  the flow of labour income received in  $t$ , the future financial wealth may be written as follows:

$$W(t + \Delta t) = W(t) + \frac{W(t) X'(t) \Delta P(t)}{P(t)} + r(1 - X(t)) W(t) - c(t) + y(t) \quad (9)$$

Future financial wealth is equal to the current financial wealth increased by both the return of investment in risky assets and the interest rate received from proportion  $1 - X(t)$  invested in the riskless asset. To this amount, we have to add current labour income flow minus current consumption expenditure.

The riskless asset is assumed to yield a known interest rate fixed at  $r$ . The yields of the risky financial assets are assumed to follow a Wiener process with the stochastic differential written as :

$$\frac{\Delta P(t)}{P(t)} = [\mu_s dt + \sigma'_s dZ(t)] \quad (10)$$

where  $\mu$  is the vector of the average returns per unit of time and  $\Sigma \equiv \sigma'_s \sigma_s$  the variances-covariances matrix of the returns of the  $N$  risky assets per unit of time, of dimension  $(N + 4) \times N$ . In addition, the returns of the  $N$  risky assets show the covariances  $\Sigma_\omega \equiv \sigma'_s \sigma_\omega$  with wage adjustments by level of education, and the covariances  $\Sigma_\theta \equiv \sigma'_s \sigma_\theta$  and  $\Sigma_\delta \equiv \sigma'_s \sigma_\delta$  with the parameters of net productivity and depreciation of human capital.

By replacing  $dP(t)$  by its value given by (10) into equation (9), variations of financial wealth between dates  $t$  and  $t + \Delta t$ , when  $\Delta t$  tends to zero may be written as

$$dW(t) = [(rW(t) + y(t) - c(t)) + W(t)(\mu - r1)'X(t)]dt + W(t)X'(t)\Gamma' dZ(t) \quad (11)$$

Equations (8) and (11) constitute the constraints under which individuals are, over the whole of their life cycle, presumed to maximise their time-separable utility function, which depends on consumption, leisure and terminal wealth<sup>10</sup>. More precisely, the programme to be resolved is posed in the following way :

$$Max E_t \int_t^T e^{-\rho(s-t)} u[c(s), l(s)] ds + e^{-\rho(T-t)} B[W(T), T] \quad (12)$$

under the constraints<sup>11</sup>:

$$\begin{aligned} \frac{dk}{k} &= [\mu_\omega + (\mu_\theta + \sigma'_\theta \sigma_\omega) \lambda - \mu_\delta - \sigma'_\delta \sigma_\omega] dt + (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' dZ \\ dW &= [rW + (1 - \lambda - l)k - c + WX(\mu - r)] dt + WX\sigma'_s dZ \\ \text{et } c &\geq 0, 0 \leq \lambda, l \leq 1 \end{aligned}$$

## 4 Optimal investment in human capital : general case

On the basis of this initial stochastic programme, Bellman's optimality principle allows us to express the following equivalences :

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<sup>10</sup>When the individual's planning horizon is not infinite, a bequest function  $B[W(T), T]$  may be added to the problem.

<sup>11</sup>In what follows, to simplify, we respectively replace  $c(t), l(t), k(t), w(t), X(t)$  and  $e(t)$ , by  $c, l, k, w, X, e$ .

$$\begin{aligned}
J(t, W, k, Y) &= e^{-\rho t} V(t, W, k, Y) \quad (13) \\
V[k, W, Y, t, T] &\equiv \text{Max } E_t \int_t^T e^{-\rho(s-t)} u[c(s), l(s)] \Delta s + e^{-\rho(T-t)} B[W(T), T] \quad (14) \\
&\equiv \text{Max } E_t \left\{ \int_t^T e^{-\rho(s-t)} u[c, l, k, t] \Delta s \right. \\
&\quad \left. + \text{Max } E_{t+\Delta t} \int_{t+\Delta t}^T e^{-\rho(s-t)} u[c, l, k, t] \Delta s + e^{-\rho(T-t)} B[W(T), T] \right\} \\
&\equiv \text{Max } E_t \left\{ \int_t^{t+\Delta t} e^{-\rho(s-t)} u[c, l, k, t] \Delta s + V[k, W, Y, t + \Delta t, T] \right\} \quad (15)
\end{aligned}$$

$V[k, W, Y, t, T]$  is called the indirect utility function. It corresponds to the maximum utility level which individuals can expect to obtain over the whole of their life cycle if they make an optimal allocation of their time and wealth to the different activities offered to them. It is presumed to be strictly increasing and concave in  $k$ ,  $W$  and  $Y$ . In this model, human and financial capital are presumed to be imperfect substitutes, which explains why  $k$  and  $W$  appear separately in the indirect utility function  $V$ . This hypothesis stems from the fact that, unlike financial capital, human capital cannot be freely bought or sold on the market. It is non-commercial and partly irreversible. As mentioned above, we introduce a vector of state variables  $Y$ , to allow intertemporal hedging against macro-economic risks, through correlations between these random process and those describing human and financial capital accumulation processes. The fundamental reason is that, if such correlations exist, then global variability of human and financial capital may partly be predictable and will be integrated into investment decisions.

Bellman's optimality principle allows us to show the present and future effects of the variations of the control variables on the optimal trajectory of the state variables. More specifically, the maximum utility obtained in the interval  $[t, T]$  results from a series of choices of control variables  $\lambda$ ,  $WX$ ,  $c$  and  $l$  and the evolution of the state variables  $k$ ,  $W$  and  $Y$ . Consequently, the first term  $u[c, l, k, t] \Delta s$  gives the value of the direct effects of the decision made at instant  $t$ , while the second  $V[k, W, Y, t + \Delta t, T]$  measures the indirect effects, namely the cumulative utility which individuals can obtain in the interval  $[t + \Delta t, T]$  given the choice made in the interval  $[t, t + \Delta t]$ .

In appendix A.2, we show that programme (15) is equivalent to :

$$\begin{aligned}
0 &\equiv \text{Max } \{ u[c, l] + V_k (\mu_\omega + \mu_\theta \lambda + \lambda \sigma'_\theta \sigma_\omega - \delta - \sigma'_\omega \sigma_\delta) k \\
&\quad + V_W [rW + (1 - \lambda - l)k - c + WX(\mu - r)] \\
&\quad + \frac{1}{2} V_{WW} W^2 X^2 \sigma_s^2 + \frac{1}{2} V_{kk} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta) k^2 \\
&\quad + V'_Y \mu_Y + \frac{1}{2} V_{YY} \sigma'_Y \sigma_Y + WX V_{WY} \sigma'_s \sigma_Y + V_{kW} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_s k WX \\
&\quad + V_{kY} k (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_Y - \rho V + V_t \} \quad (16)
\end{aligned}$$

This equation with stochastic partial derivatives is known as a Bellman stochastic optimal control equation or a Hamilton-Jacobi-Bellman equation. Its internal solutions are obtained in the classic way, by positing each of the partial derivatives equal to zero<sup>12</sup>. We thus obtain the optimality conditions which implicitly define the four solutions  $c^*(t)$ ,  $l^*(t)$ ,  $wX^*(t)$ , and  $\lambda^*(t)$ .

The first two conditions, for  $0 \leq t \leq T$ , are immediate :

$$\frac{u_c}{V_W} = 1 \quad (17)$$

and

$$\frac{u_c}{u_l} = \frac{1}{k} \quad (18)$$

Conditions (17) and (18) are similar to those of models with certainty since the risk-related parameters are not involved in the determination of these two optimality conditions. Furthermore, current consumption and current leisure time are affected in a way which is similar to the certainty case by the increments in financial wealth and human capital. In effect, a marginal increment in financial wealth increases the individual's current consumption and current leisure time<sup>13</sup>. If, like Williams, we assume that the marginal rate of substitution between consumption and leisure is independent of human capital, an increase in current human capital increases the individual's current consumption but reduces current leisure<sup>14</sup>. Indeed, the increment in human capital gives rise to a relative increase in the value of current financial wealth and thus, through (17), increases consumption. However, by increasing the marginal utility of financial wealth, the increment in human capital increases the opportunity cost of leisure and individuals are thus encouraged to reduce their leisure time in favour of work.

The third condition implies that the partial derivative of (16) with respect to  $WX$  is equal to 0, which yields :

$$V_W(\mu - r) + V_{WW}WX\sigma'_s\sigma_s + V_{WY}\sigma'_s\sigma_Y + V_{kW}(\sigma_\omega + \lambda\sigma_\theta - \sigma_\delta)' \sigma_s k = 0$$

Thus

$$X^* = -\frac{V_W}{V_{WW}W} \frac{\mu - r}{\sigma'_s\sigma_s} - \frac{V_{kW}}{V_{WW}} \frac{k}{W} \frac{(\sigma_\omega + \lambda\sigma_\theta - \sigma_\delta)' \sigma_s}{\sigma'_s\sigma_s} - \frac{V_{WY}}{V_{WW}W} \frac{\sigma'_s\sigma_Y}{\sigma'_s\sigma_s} \quad (19)$$

<sup>12</sup>The subscripts designate the partial derivatives relative to the variables.

<sup>13</sup>In fact, if  $w$  increases, then, according to (17),  $\frac{\partial u_c}{\partial W} = V_{WW}$ . Thus  $\frac{\partial c}{\partial W} = \frac{V_{WW}}{u_{cc}} > 0$ , according to the concavity hypothesis of utility functions. By applying the same principle, we also shows that leisure increases with financial wealth :  $u_l = ku_c = kV_W$ . Thus  $\frac{\partial u_l}{\partial W} = kV_{WW}$  and  $\frac{\partial l}{\partial W} = k \frac{V_{WW}}{u_{ll}} > 0$ .

<sup>14</sup>The hypothesis  $\frac{\partial}{\partial k} \left( \frac{u_c}{u_l} \right) = 0$  implies that  $\frac{\partial c}{\partial k} = -\frac{u_c}{u_l} \frac{u_{cc}}{u_{ll}} \frac{\partial l}{\partial k}$ . Utility is a strictly increasing and concave function of  $c$  and  $l$ , thus :  $u_c > 0, u_l > 0, u_{cc} < 0$  and  $u_{ll} < 0$ . We deduce that  $\partial c / \partial k$  and  $\partial l / \partial k$  are of opposite sign. However, from (17),  $\frac{\partial u_c}{\partial k} = V_{Wk}$ , thus  $\frac{\partial c}{\partial k} = \frac{V_{Wk}}{u_{cc}} > 0$ . We can conclude that  $\frac{\partial l}{\partial k} < 0$ .

Condition (19) is similar to the optimality condition of a standard mean-variance portfolio problem. In effect, it establishes that the individual's risky asset portfolio is constituted at the optimum of the three basic portfolios. The first portfolio,  $\frac{\mu - r}{\sigma'_s \sigma_s}$ , is the standard market portfolio. It is a speculative component. It represents the excess of return expected of investment in the  $N$  risky assets relative to the riskless asset. The other portfolios  $\frac{(\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_s}{\sigma'_s \sigma_s}$  and  $\frac{\sigma'_s \sigma_Y}{\sigma'_s \sigma_s}$  are composed solely of risky assets and respectively represent the intertemporal hedging component against human capital and against macroeconomic state variables.

We may note that the composition of the optimal risky asset portfolio is crucially dependent on the individuals' perception of the risk ( $\sigma'_s \sigma_s$ ), but also on the behaviour adopted in face of the risk. Investment in risky assets is thus proportional to the risk premium ( $-\frac{1}{2} \sigma'_s \sigma_s \frac{V_{WW} W}{V_W}$ ) and inversely proportional to the Arrow-Pratt measure of absolute risk aversion<sup>15</sup> : ( $-\frac{V_{WW} W}{V_W}$ ), as well as to the variance of the returns ( $\sigma'_s \sigma_s$ ). It should also be noted that the weight attached to the market portfolio is strictly the inverse of the measure of relative risk aversion.

Finally, the optimal level of human capital investment is given by :

$$V_k \mu_\theta k + V_k \sigma'_\theta \sigma_\omega k - V_W k + k^2 V_{kk} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_\theta + V_{kW} \sigma'_\theta \sigma_s k W X + V_{kY} k \sigma'_\theta \sigma_Y = 0$$

Which, if we rearrange the terms, yields :

$$\lambda^* = \left( -\frac{V_k}{V_{kk} k} \right) \frac{1}{\sigma'_\theta \sigma_\theta} \left( \mu_\theta + \sigma'_\theta \sigma_\omega - \frac{V_W}{V_k} \right) + \frac{\sigma'_\delta \sigma_\theta - \sigma'_\omega \sigma_\theta}{\sigma'_\theta \sigma_\theta} + \left( -\frac{V_k}{V_{kk} k} \right) \frac{V_{kY}}{V_k} \frac{\sigma'_\theta \sigma_Y}{\sigma'_\theta \sigma_\theta} + \left( -\frac{V_k}{V_{kk} k} \right) \frac{V_{kW}}{V_k} \frac{\sigma'_\theta \sigma_s}{\sigma'_\theta \sigma_\theta} W X \quad (20)$$

As optimal investment in risky assets, optimal investment in human capital exhibits a speculative component and hedging components. The first term on the left side of (20) is the market price of the risk bearing on human capital. It is positive. It represents the excess of return expected of human capital investment, on the labour market. The second term is a minimum-variance component. It represents a hedging component against risks inherent to the human capital accumulation process, which is independent of individual preferences. The third term is an intertemporal hedging component against macroeconomic shocks coming from the presence of state variables, which may influence stochastic processes underlying human and financial wealth dynamics. The last term is the intertemporal hedging component against financial risky assets. This last term exists if  $\sigma'_\theta \sigma_s \neq 0$ . That means, financial market shocks influence education choices. If this covariance is positive, then individual reduces his optimal investment into human capital. If this covariance is negative, the human capital demand increases to hedge against the risk on financial market. Following this way, leads to a very complex theoretical exercise. Indeed, plugging optimal value of  $X^*$  (equation 19) into (20), we obtain a very complicated structure of  $\lambda^*$ , from which it is very difficult to study the effect of each source of uncertainty. Thus, to simplify, we

<sup>15</sup>For a detailed presentation of the different measures of risk aversion, cf. Laffont (1991).

suppose in the subsequent analysis that  $\sigma'_\theta \sigma_s = 0$ . In other words, as Williams (1979), we suppose that chocs on the financial market have no effect on  $\theta$ , the efficiency of training. This is a weak assumption and rather realistic<sup>16</sup>.

The optimal level of human capital investment is now :

$$\lambda^* = \left( -\frac{V_k}{V_{kk}k} \right) \frac{1}{\sigma'_\theta \sigma_\theta} \left( \mu_\theta + \sigma'_\theta \sigma_\omega - \frac{V_W}{V_k} \right) + \frac{\sigma'_\delta \sigma_\theta - \sigma'_\omega \sigma_\theta}{\sigma'_\theta \sigma_\theta} + \left( -\frac{V_k}{V_{kk}k} \right) \frac{V_{kY}}{V_k} \frac{\sigma'_\theta \sigma_Y}{\sigma'_\theta \sigma_\theta} \quad (21)$$

Such a structure of optimal time allocated to education fundamentally depends on the perception of the different risks, as represented by the whole of the instantaneous variances and covariances. It depends directly on the risk associated with learning efficiency, measured by the variance of the marginal product of education  $\sigma'_\theta \sigma_\theta$ . On the other hand, all other risks intervene indirectly through their intermediary of the covariances  $\sigma'_\delta \sigma_\theta$ ,  $\sigma'_\omega \sigma_\theta$ , and  $\sigma'_\theta \sigma_Y$  with the marginal product of human capital  $\theta$ .

Therefore, the structure of optimal demand of human capital, and the effect of risks we will in-deph study next section, crucially depends on the parameter  $\theta$ .

The main difference from the Williams (1979) model is that here, the overall effect of the different sources of uncertainty on human capital investment cannot be unambiguously determined, because it depends on the sign and scale of the covariances. In fact, the absence of state variables and the hypothesis of independence between the risk over wage rates and risks existing in the training process, and also the assumption of independence of the financial market, lead Williams to make the following predictions: the increase of risk over the marginal product of education results in a decrease in human capital investment and earnings over the life cycle. Similarly, assuming a negative relationship between learning efficiency and depreciation ( $\sigma_{\theta\delta} < 0$ ), individuals are encouraged to reduce their investment to protect themselves against an increased risk of skills depreciation.

In our model, these findings remain partly valid but they are insufficient to characterise the overall effect of risk because other risks appear in the optimal solution in the general case. In order to measure the impact of a variation in these types of risk, it is necessary to specify individual preferences in greater detail because the sign of risks' effect depend fundamentally on the Arrow-Pratt measure of relative risk aversion<sup>17</sup>  $-\frac{V_{kkk}}{V_k}$ . But this index, constructed from the indirect utility function  $V$ , generally absorbs the effect of all the parameters of the model over the time structure of the optimal values. Without additional restrictions on individual preferences, it is not possible to obtain explicit solutions for (21) nor to evaluate the impact of the variations in the model's parameters over the life cycle. This is why we specify the individual preferences in greater detail in the following section. Moreover,  $Y$  is a vector of state variables, so it is impossible to describe all correlations between  $\theta$  and each state variable, and

<sup>16</sup>We might think that a relation between risky assets' price and education efficiency is difficult to empirically justify.

<sup>17</sup>In particular, any increase in risk aversion results, as expected, in a reduction of investment in human capital.

their effect on the decision to invest in human capital<sup>18</sup>. That's why in next section, we focus on one macroeconomic state variable, which makes sense for the labour economist and which capture a real source of variability of earnings over the life cycle : the unemployment risk.

## 5 Logarithmic preferences, unemployment, and effect of uncertainties

Jusqu'ici, nous avons supposé que les individus partageaient leur temps entre activités de travail, d'éducation et de loisir. Nous supposons maintenant que les individus peuvent être au chômage une partie de leur temps. Nous notons  $u(t)$ , la partie du temps déduite du temps de travail pendant laquelle les individus sont au chômage. Sous cette nouvelle hypothèse, le revenu courant du travail devient

$$y(t) = (1 - \lambda(t) - l(t) - u(t))k(t) \quad (22)$$

Où  $k(t)$  la valeur du capital humain accumulé par l'individu à la date  $t$  mesure toujours le revenu du travail correspondant au temps de travail durant la période courante  $(1 - \lambda - l - u)$ .

Suppose now that current unemployment is perfectly observed but future unemployment rate is unknown, and is driven by a diffusion process of the following form :

$$du(t) = \mu_u dt + \sigma_u dZ(t) \quad (23)$$

Using formula (23) and replacing (22) into (11), the dynamics of financial wealth, after applying Itô's lemma, take a new form :

$$dW = [rW + (1 - \lambda - l - \mu_u)k - c + WX(\mu - r) - kWX\sigma'_s\sigma_u] dt + [WX\sigma'_s - k\sigma_u] dZ \quad (24)$$

Let us assume that individual preferences are described by a logarithmic instantaneous utility function. In this modified setting, the program which the individual is submitted is :

$$\text{Max } E_t \int_t^T e^{-\rho(s-t)} [\phi_c \ln c + (1 - \phi_c) \ln l] ds + e^{-\rho(T-t)} B[W(T), T] \quad (25)$$

Under the constraints:

$$\begin{aligned} \frac{dk}{k} &= (\mu_\omega + \mu_\theta \lambda - \mu_\delta + \lambda \sigma'_\theta \sigma_\omega - \sigma'_\delta \sigma_\omega) dt + (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' dZ \\ dW &= [rW + (1 - \lambda - l - \mu_u)k - c + WX(\mu - r) - kWX\sigma'_s\sigma_u] dt + [WX\sigma'_s - k\sigma_u] dZ \\ \text{et } c &\geq 0, 0 \leq \lambda, l \leq 1 \end{aligned} \quad (26)$$

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<sup>18</sup>Moreover, depending on macroeconomic variables, the sign of  $V_{kY}$  may be positive or negative. The analysis is all the more complex.



As demonstrated in appendix A.3, the resolution of problem (25) under constraints (26) plus appropriate transversality conditions provide the following explicit solutions :

$$c^* = \frac{\phi_c}{A(t, T)} [B(t, T)k + W - C(t, T)u] \quad (27)$$

$$l^* = \frac{1 - \phi_c}{A(t, T)\mu_u} \left[ B(t, T) + \frac{W - C(t, T)u}{k} \right] \quad (28)$$

$$WX^* = [B(t, T)k + W - C(t, T)u] \frac{(\mu - r) - \sigma'_s \sigma_u}{\sigma'_s \sigma_s} + \frac{\sigma'_s \sigma_u}{\sigma'_s \sigma_s} k + \frac{\sigma'_s \sigma_u}{\sigma'_s \sigma_s} C(t, T) - \frac{(\sigma_\omega - \sigma_\delta)' \sigma_s}{\sigma'_s \sigma_s} B(t, T)k \quad (29)$$

$$\lambda^* = \left[ 1 + \frac{W - C(t, T)u}{B(t, T)k} \right] \frac{1}{\sigma'_\theta \sigma_\theta} \left( \mu_\theta + \sigma'_\theta \sigma_\omega - \frac{1}{B(t, T)} \right) - \frac{(\sigma'_\omega \sigma_\theta - \sigma'_\delta \sigma_\theta)}{\sigma'_\theta \sigma_\theta} + \frac{1 + C(t, T)}{B(t, T)} \frac{\sigma'_\theta \sigma_u}{\sigma'_\theta \sigma_\theta} \quad (30)$$

Both optimal consumption (27) and leisure (28) are increasing function financial wealth and decreasing in current unemployment, as expected. They also increase with their respective weight in the utility function. Unlike optimal consumption, optimal time devoted to leisure depends on future unemployment. More precisely, expected future unemployment  $\mu_u$  reduce leisure time. This comes from the fact that future unemployment reduces life time earnings, so that individual has an incentive to work more today. The same trade-off explains why the current level of human capital increases consumption but decrease optimal leisure. Indeed, human capital is the wage associated with each unit of time worked. So, it increases the labor income devoted to current consumption, but it increases also the opportunity cost of leisure. that's why leisure is reduced with wage and unemployment.

As in the general case, investment in risky assets (29), increases with current financial wealth  $W(t)$ , and decreases with increments in the absolute risk aversion index  $\frac{1}{B(t, T)k + W - C(t, T)u}$ . As expected, current unemployment, by reducing current labour earning, reduces also the proportion of financial wealth which may be invested in risky assets. The effect of current human capital is not clear-cut. On one hand, it always increases the speculative component, but, on the other hand it ambiguously affects hedging components against unemployment and human capital, through covariances between the risky assets' price and other stochastic processes.

Before in-depth studying the effects of risks, we may notice that, as in the certainty case, optimal investment in human capital (30) is a decreasing function of the current level of human capital. It is important to note that, such a result lies on the decreasing return assumption in the certainty case. Here, decreasing investment in human capital comes from the resolution under uncertainty, which does not force us to presuppose a decreasing productivity of human capital production.

On the other hand, any increase in current financial wealth increases the human capital investment. In other words, the model predicts that richer individuals invest more in education, which is often observed empirically. This fundamental property, brought out by Williams (1979) and found here once again, sharply contrasts with the predictions of models without uncertainty. Indeed, in Heckman's model, the structure of which is similar to ours, human capital investment is not affected by variations in financial wealth (*cf.* Heckman 1976, p. 523). As a result, the observed differences in education levels between individuals with different financial resources are commonly attributed to differences in real interest rates. The latter are interpreted as the indirect proof of an imperfection in the capital market. It is likely, as Becker (1993) already pointed out, that this heterogeneity in individual interest rates exists and that it increases the financial constraint of the least wealthy individuals. In our case, however, the heterogeneity of the opportunity cost of education does not stem from the heterogeneity of the interest rate. Other things being equal, individuals with limited financial resources attain levels of education, and thus of wages, which are lower than those of those with greater financial resources because they are confronted with a higher opportunity cost of human capital. This theoretical result, which is fairly realistic, is obtained independently of any imperfection of the capital market.

When education is considered as a risky asset, the structure of optimal human capital investment (30) exhibits a speculative component and hedging components, as investment on the financial market. The first component represents the excess of return expected on labour market of risky investment in human capital. In other words, it is the payment the individual expects from his risk taking. This component incorporate the relative risk aversion index  $\frac{1}{1 + \frac{W - C(t,T)u}{B(t,T)k}}$ , which, as in the general case, always decreases optimal investment. In the same way, the marginal rate of substitution between human capital and financial capital  $\frac{1}{B(t,T)}$  reduce time allocated to education. During time, marginal value of human capital decreases relative to financial capital. Contrary to human capital, financial capital may be transmitted to future generation at the end of the life cycle. Marginal value of wealth may have a positive value whereas marginal value of human capital tend to zero at the end of the life cycle. This explain why human capital is concentrated at the begening of the life cycle and decrease during time.

We can see too, that current unemployment mechanically reduce optimal investment. Subtly, by using the expression of optimal leisure time, we can show that future unknown unemployment affects current human capital investment<sup>19</sup>. Indeed, an increase of future unemployment mean increases current time allocated to education. As we have already explained, future unemployment reduces the opportunity cost of human capital investment, that is the labour income the individual gives up when he spends time to education activities. So education is encouraged. On the other hand, future unemployment variability, reduce investment in human capital. If  $\sigma'_\theta \sigma_u < 0$ , the intertemporal hedging component

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<sup>19</sup>Replacing expression  $[B(t,T)k + W - C(t,T)u]$  from (27) into (30) give :

$$\lambda^* = \left[ \frac{A(t,T)\mu_u l^*}{1 - \phi_c} \right] \frac{1}{\sigma'_\theta \sigma_\theta} \left( \mu_\theta + \sigma'_\theta \sigma_\omega - \frac{1}{B(t,T)} \right) - \frac{(\sigma'_\omega \sigma_\theta - \sigma'_\delta \sigma_\theta)}{\sigma'_\theta \sigma_\theta} + \frac{1 + C(t,T)}{B(t,T)} \frac{\sigma'_\theta \sigma_u}{\sigma'_\theta \sigma_\theta}$$

against unemployment risk is négative, reducing optimal investment. This is the most likely case, since a negative correlation between  $\theta$  and  $u$  means that the most able individuals are those who reach lower levels of unemployment<sup>20</sup>. In a same manner, the mean efficiency of education increase investment, but risk concerning efficiency, measured by the variance  $\sigma'_\theta \sigma_\theta$ , reduces human capital investment. the risk bearing on depreciation of human capital reduces current investment in education too. Thus, facing existing risks, individuals use human capital investment to cover themselves and generally reduce their investment. On this point, our result join those of Williams (1979). But, despite the negative sign of hedging components, it is not sufficient to conclude that uncertainty always decreases human capital investment. Indeed, as we can see, wage risk intervenes in both minimum variance component and speculative component with two opposit effects. We show that the net effect is positive :

$$\frac{\partial \lambda}{\partial \sigma'_\omega \sigma_\theta} = \frac{1}{\sigma'_\theta \sigma_\theta} \frac{W - C(t, T) u}{B(t, T) k} > 0^{21}$$

Facing an increase in wage risk, the individual invest more in human capital.

Our economic interpretation of this results is the following. When education is considered as a risky asset, the individual exposes itself to risk when he invests. He takes the risk to see his investment not remunerated. To cover itself against this uncertainty, he naturally reduces his investment except if it exists a sufficiently high risk premium. It is the case for the wage risk. In the model, such a compensation is possible because the price of human capital enters in the dynamic of human capital accumulation. This not the case for unemployment risk, which is really exogenous to the human capital accumulation process. That's why it only comes into the intertemporal hedging component with a negative effect. No risk premium involves no risk taking, so that individuals allocate more time to work.

These novel results are in sharp contrast with the clear-cut conclusions of Williams (1979) concerning the effect of risk on human capital investment. The extension which we propose here permits a better understanding of why the effect of aggregate uncertainty is difficult to establish without ambiguity. The reason is that different sources of uncertainty may have contradictory effects on investment. The uncertainty about the process of human capital accumulation exercises a negative direct effect on the human capital investment while the risk over the labour market may indirectly encourage that investment. The perception individuals have of these different risks, expressed by the extent of the different covariances, is fundamental in the decision to invest in human capital.

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<sup>20</sup>The opposit assumption would be absurd.

<sup>21</sup>Indeed,  $W > C(t, T) u$  otherwise the value function  $V(\cdot)$  is not defined when  $k$  tends to zero.

## 6 Conclusion

This study has as its starting point a critique of the standard human capital model, which is no longer capable of describing a certain number of realities observed in the majority of developed countries. The existence (to varying degrees depending on the country and the immediate economic situation) of the phenomena of over-education and unemployment, which have been particularly marked among young people for several years, no longer allows decisions in the area of educational policy to be based on a model assuming that individuals have a perfect knowledge of their future situation at the time they make their decisions about education.

In order to go beyond this critique, we have presented a theoretical model which situates the analysis of the demand for education in a context of uncertainty. The hypothesis defended in this study is that the effect of uncertainty cannot be determined *ex ante* without ambiguity if the different sources of uncertainty, bearing at once on the learning process and the future labour-market situation, are not identified in detail. For this reason, we have proposed a general frame of analysis built on Williams (1979) contribution.

To reflect the problems observed on the European labour markets, we have integrated the risk of future unemployment, as a state variable. In addition, by removing the hypothesis of independence between variables related to the labour market and those defining the process of human capital accumulation, we have been able to show the fundamental role of individuals' perception of the future conditions of price of human capital, which are integrated and priced in the decision to invest. But, if such a risk premium, can not be expected by individuals, then they reduce investment in education to protect themselves against risks. Such results lies fundamentally on the hypothesis that human capital is a risky asset. That is when an individual invests in education, the expose himself to risk, because education increases by assumption future earnings variability. Other hypothesis exist in the literature about le role of education. The most famous one is that would play the role of a signal. That is education do not increase human capital, but it reveals during time the real ability of students. In other word, higher levels of education reduce uncertainty about real ability. A natural, but original, extension of this work would be to investigate this assumption, insofar as we have seen the key determinant of our results is how education come into the dynamic of labour earnings.

## 7 References

**Anderberg D. & Andersson F., (2003)**, "Investment in human capital, wage uncertainty and public policy", *Journal of Public Economics*, **87**, pp 1521-1537.

**Becker G., (1993)**, *Human Capital : A Theoretical and Empirical Analysis*. 3rd edition. Columbia University Press.

**Ben-Porath Y., (1967)**, “The production of human capital and the life cycle of earnings”. *Journal of Political Economy*, **75**, 352-365.

**Card D., (2001)**, “Estimating the return to schooling : progress on some persistent econometric problems”. *Econometrica*, **69** (5), 1127-1160.

**Dixit A. K., (1993)**, *The Art of Smooth Pasting*. Harwood Academic Publisher.

**Dixit A. K., & Pindyck R. S., (1994)**, *Investment under uncertainty*. Princeton University, Princeton , New Jersey.

**Fan C., (1993)**, “Schooling as a job search process”, *Economics Letters*, **41**, 85-91.

**Griliches Z., (1977)**, “Estimating the return to schooling : some econometric problems”, *Econometrica*, **45**, 1-22

**Heckman J., (1976)**, “A life cycle model of earnings learning and consumption”. *Journal of Political Economy*, **84**, 511-544.

**Hogan V., & Walker I., (2002)**, “ Education choice under uncertainty”. *mimeo*, University College Dublin, presented to the *Royal Economic Society Annual Conference 2002*.

**Kamien M., & Schwartz N., (1991)**, *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*. 2nd edition, North-Holland

**Kodde D., (1986)**, “Uncertainty and the demand for education”. *The Review of Economics and Statistics*, **68** (3), 460-467.

**Laffont J-J., (1991)**, *Economie de l'Information et de l'Incertain*. Economica.

**Low S. & Ormiston M., (1991)**, ”Stochastic Earnings Functions, Risk, and the Rate of Return to Schooling”, *Southern Economic Journal*, Apr91, **57** (4), 1124-1133.

**Lehvari D., & Weiss Y., (1974)**, “The effect of risk on the investment in human capital ”. *American Economic Review*, **64** (6), 950-963.

**Merton R., (1971)**, “Optimum consumption and portfolio rules in a continuous-time model”. *Journal of Economic Theory*, **3**, 373-413.

**Mincer J., (1974)**, *Schooling Experience and Earnings*. New-York: National Bureau of Economic Research.

**Moulet S., (2000)**, “Contributions à l'analyse des rendements éducatifs : Enjeux théoriques et problèmes empiriques d'identification”, *Thèse de Doctorat*, Université de la Méditerranée, Décembre.

**Roger P., (1991)**, *Les outils de la modélisation financière*. Presses Universitaires de France.

**Rothschild M & Stiglitz J. E., (1970)**, “Increasing risk: I. A definition”, *Journal of Economic Theory*. **2**, 225-243.

**Rothschild M & Stiglitz J. E., (1971)**, “Increasing risk: II. Its economic consequences”. *Journal of Economic Theory*, **3**, 66-84.

**Sloane P. J., (2002)**, ‘Much ado about nothing? What does the over-education literature really tell us?’, Keynote address, *International Conference on Over-Education in Europe: What Do We Know?*, Berlin 22-23 November 2002.

**Snow A. & Warren R. S., (1990)**, “Human capital investment and labour supply under uncertainty”. *International Economic Review*, **31** (1), 195-206.

**Weiss Y., (1986)**, “The determination of life cycle earnings : a survey”. *Handbook of Labor Economics*. **1**, chap. 11, 603-640.

**Williams J. T., (1979)**, “Uncertainty and the accumulation of human capital over the life cycle”. *Journal of Business*, **52** (2), 521-548.

**Willis R. J., (1986)**, “Wage determinants: a survey and reinterpretation of human capital earnig functions”, *Handbook of Labor Economics*, **1**, chap. 10, 525-602.

**Willis R. J., & Rosen S., (1979)**, “Education and Self-Selection”, *Journal of Political Economy*, **87**, 7-35.

## A Appendix : Resolution of the stochastic optimal control problem using continuous time dynamic programming techniques

### A.1 Proof of the stochastic equation (8) of human capital accumulation:

The second order Taylor expansion of equation (4) has the following form :

$$\begin{aligned}
k(t + \Delta t) = & k(t) + \frac{\partial k(t + \Delta t)}{\partial t} \Delta t + \frac{\partial k(t + \Delta t)}{\partial \omega(t + \Delta t)} \Delta \omega + \frac{\partial k(t + \Delta t)}{\partial \theta(t, t + \Delta t)} \Delta \theta \\
& + \frac{\partial k(t + \Delta t)}{\partial \delta(t, t + \Delta t)} \Delta \delta + \frac{1}{2} \frac{\partial^2 k(t + \Delta t)}{\partial \omega^2(t + \Delta t)} (\Delta \omega)^2 + \frac{1}{2} \frac{\partial^2 k(t + \Delta t)}{\partial \theta^2(t, t + \Delta t)} (\Delta \theta)^2 \\
& + \frac{1}{2} \frac{\partial^2 k(t + \Delta t)}{\partial \delta^2(t, t + \Delta t)} (\Delta \delta)^2 + \frac{\partial^2 k(t + \Delta t)}{\partial \omega(t + \Delta t) \partial \theta(t, t + \Delta t)} (\Delta \omega \Delta \theta) \\
& + \frac{\partial^2 k(t + \Delta t)}{\partial \omega(t + \Delta t) \partial \delta(t, t + \Delta t)} (\Delta \omega \Delta \delta) + \frac{\partial^2 k(t + \Delta t)}{\partial \theta(t, t + \Delta t) \partial \delta(t, t + \Delta t)} (\Delta \theta \Delta \delta) \\
& + \frac{\partial^2 k(t + \Delta t)}{\partial t^2} (\Delta t)^2 + \frac{\partial^2 k(t + \Delta t)}{\partial t \partial \omega(t + \Delta t)} (\Delta t \Delta \omega) \\
& + \frac{\partial^2 k(t + \Delta t)}{\partial t \partial \theta(t, t + \Delta t)} (\Delta t \Delta \theta) + \frac{\partial^2 k(t + \Delta t)}{\partial t \partial \delta(t, t + \Delta t)} (\Delta t \Delta \delta) + R(\Delta t)
\end{aligned}$$

By noting that terms in  $\Delta t$  of order 2 and beyond are infinitely "small" compared with those in  $\Delta t$ , they will be systematically neglected in calculations. The last six terms thus disappear from the above development. While also noting that  $\frac{\partial^2 k(t + \Delta t)}{\partial \omega^2} = \frac{\partial^2 k(t + \Delta t)}{\partial \theta^2} = \frac{\partial^2 k(t + \Delta t)}{\partial \delta^2} = 0$ , the Taylor series can be rewritten in the following way :

$$\begin{aligned}
k(t + \Delta t) = & k(t) + \frac{\partial k(t + \Delta t)}{\partial t} \Delta t + \frac{\partial k(t + \Delta t)}{\partial \omega(t + \Delta t)} \Delta \omega + \frac{\partial k(t + \Delta t)}{\partial \theta(t, t + \Delta t)} \Delta \theta \\
& + \frac{\partial k(t + \Delta t)}{\partial \delta(t, t + \Delta t)} \Delta \delta + \frac{\partial^2 k(t + \Delta t)}{\partial \omega(t + \Delta t) \partial \theta(t, t + \Delta t)} (\Delta \omega \Delta \theta) \\
& + \frac{\partial^2 k(t + \Delta t)}{\partial \omega(t + \Delta t) \partial \delta(t, t + \Delta t)} (\Delta \omega \Delta \delta) + \frac{\partial^2 k(t + \Delta t)}{\partial \theta(t, t + \Delta t) \partial \delta(t, t + \Delta t)} (\Delta \theta \Delta \delta)
\end{aligned}$$

$t$  do not directly intervenes as a dependent variable in the expression of  $k(t + \Delta t)$ , which implies that  $\frac{\partial k(t + \Delta t)}{\partial t} \Delta t = 0$ . The calculation of partial derivatives and the substitution of  $\Delta \omega, \Delta \theta$  and  $\Delta \delta$  by their respective value (5), (6), (7), (8) give :

$$\begin{aligned}
k(t + \Delta t) - k(t) &= \frac{1}{\omega(t)} [1 - \delta(t, t + \Delta t) + \theta(t, t + \Delta t)\lambda(t)] k(t) \omega(t) (\mu_\omega \Delta t + \sigma'_\omega \Delta Z(t)) \\
&\quad + \frac{\omega(t + \Delta t)}{\omega(t)} \lambda(t) k(t) (\mu_\theta \Delta t + \sigma'_\theta \Delta Z(t)) - \frac{\omega(t + \Delta t)}{\omega(t)} k(t) (\mu_\delta \Delta t + \sigma'_\delta \Delta Z(t)) \\
&\quad + \frac{1}{\omega(t)} \lambda(t) k(t) \omega(t) (\mu_\omega \Delta t + \sigma'_\omega \Delta Z(t)) (\mu_\theta \Delta t + \sigma'_\theta \Delta Z(t)) \\
&\quad - \frac{1}{\omega(t)} k(t) \omega(t) (\mu_\omega \Delta t + \sigma'_\omega \Delta Z(t)) (\mu_\delta \Delta t + \sigma'_\delta \Delta Z(t))
\end{aligned}$$

By simplifying by  $\omega(t)$ , at the first, third, and fourth line; by putting  $k(t)$  in factor and passing it from the left side of the equation; and by developing the last five terms, one obtains :

$$\begin{aligned}
\frac{\Delta k(t)}{k(t)} &= [1 - \delta(t, t + \Delta t) + \theta(t, t + \Delta t)\lambda(t)] (\mu_\omega \Delta t + \sigma'_\omega \Delta Z(t)) \\
&\quad + \frac{\omega(t + \Delta t)}{\omega(t)} \lambda(t) (\mu_\theta \Delta t + \sigma'_\theta \Delta Z(t)) - \frac{\omega(t + \Delta t)}{\omega(t)} (\mu_\delta \Delta t + \sigma'_\delta \Delta Z(t)) \\
&\quad + \lambda(t) \left[ \mu_\omega \mu_\theta (\Delta t)^2 + \mu_\omega \sigma'_\theta \Delta Z(t) \Delta t + \mu_\theta \sigma'_\omega \Delta Z(t) \Delta t + \rho_{\omega\theta} \sigma'_\omega \sigma_\theta (\Delta Z(t))^2 \right] \\
&\quad - \left[ \mu_\omega \mu_\delta (\Delta t)^2 + \mu_\omega \sigma'_\delta \Delta Z(t) \Delta t + \mu_\delta \sigma'_\omega \Delta Z(t) \Delta t + \rho_{\delta\omega} \sigma'_\delta \sigma_\omega (\Delta Z(t))^2 \right]
\end{aligned}$$

where  $\rho$  denote the instantaneous correlation between stochastic processes.

When  $\Delta t$  tends to 0 :  $\frac{\omega(t + \Delta t)}{\omega(t)} = 1$  and  $[1 - \delta(t, t + \Delta t) + \theta(t, t + \Delta t)\lambda(t)] = 1$  from equation (23), thus :

$$\begin{aligned}
\frac{\Delta k(t)}{k(t)} &= (\mu_\omega dt + \sigma'_\omega dZ(t)) + \lambda(t) (\mu_\theta dt + \sigma'_\theta dZ(t)) - (\mu_\delta dt + \sigma'_\delta dZ(t)) \\
&\quad + \lambda(t) \left[ \mu_\omega \mu_\theta (dt)^2 + \mu_\omega \sigma'_\theta dZ(t) dt + \mu_\theta \sigma'_\omega dZ(t) dt + \rho_{\omega\theta} \sigma'_\omega \sigma_\theta (dZ(t))^2 \right] \\
&\quad - \left[ \mu_\omega \mu_\delta (dt)^2 + \mu_\omega \sigma'_\delta dZ(t) dt + \mu_\delta \sigma'_\omega dZ(t) dt + \rho_{\delta\omega} \sigma'_\delta \sigma_\omega (dZ(t))^2 \right]
\end{aligned}$$

Finally, from properties of the standard Wiener process:  $(dt)^2 = dZ(t) dt = o(dt)$  and  $(dZ(t))^2 = dt + o(dt)$ . When rearranging terms, we find the final expression of the stochastic equation of human capital accumulation (8) :

$$\begin{aligned}
\frac{dk(t)}{k(t)} &= (\mu_\omega + (\mu_\theta + \sigma_{\theta\omega}) \lambda(t) - \mu_\delta - \sigma_{\delta\omega}) dt \\
&\quad + (\sigma_\omega + \sigma_\theta \lambda(t) - \sigma_\delta)' dZ(t)
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{\theta\omega} &= \rho_{\omega\theta} \sigma'_\theta \sigma_\omega = Cov(\omega, \theta) \\
\sigma_{\delta\omega} &= \rho_{\delta\omega} \sigma'_\delta \sigma_\omega = Cov(\delta, \omega)
\end{aligned}$$



## A.2 Proof of equation (16)

The first term of (15) in the square brackets can be approximated by  $u[c, l, t] \Delta t$ . Applying to  $V[k, W, Y, t + \Delta t, T]$  the second order Taylor expansion, and neglecting terms of higher order <sup>22</sup>, then we obtain :

$$\begin{aligned} V[k(t), w(t), T] &\equiv \text{Max } E_t \{u[c, l, t] \Delta t + V[k, W, T] \\ &+ V_t \Delta t + V_k \Delta k + V_W \Delta W + V_Y \Delta Y + \frac{1}{2} (\Delta k)' V_{kk} \Delta k + \frac{1}{2} (\Delta W)' V_{WW} \Delta W \\ &+ \frac{1}{2} (\Delta Y)' V_{YY} \Delta Y + V_{kW} (\Delta k)' \Delta W + V_{kY} (\Delta k)' \Delta Y + V_{YW} (\Delta Y)' \Delta W \} \end{aligned}$$

By simplifying by  $V[k, W, T]$  in each member of the equation and applying Itô's lemma when  $\Delta t$  tends to 0, we find :

$$\begin{aligned} 0 &\equiv \text{Max } E_t \{u[c, l, t] dt + \left[ V_t + \mu_k V_k + \mu_W V_W + \mu_Y V_Y + \frac{1}{2} \sigma_k^2 V_{kk} + \frac{1}{2} \sigma_W^2 V_{WW} \right. \\ &\quad \left. + \frac{1}{2} \sigma_Y^2 V_{YY} + \sigma'_k \sigma_W V_{kW} + \sigma'_k \sigma_Y V_{kY} + \sigma'_Y \sigma_W V_{YW} \right] dt \\ &\quad + [\sigma_k V_k + \sigma_W V_W + \sigma_Y V_Y] dZ(t) \} \end{aligned} \quad (31)$$

If we define

$$\begin{aligned} dV &= \left[ V_t + \mu_k V_k + \mu_W V_W + \mu_Y V_Y + \frac{1}{2} \sigma_k^2 V_{kk} + \frac{1}{2} \sigma_W^2 V_{WW} \right. \\ &\quad \left. + \frac{1}{2} \sigma_Y^2 V_{YY} + \sigma'_k \sigma_W V_{kW} + \sigma'_k \sigma_Y V_{kY} + \sigma'_Y \sigma_W V_{YW} \right] dt \\ &\quad + [\sigma_k V_k + \sigma_W V_W + \sigma_Y V_Y] dZ(t) \} \end{aligned} \quad (32)$$

and expand the conditional expectation operator, we obtain the following *stochastic partial derivative equation* (SPDE) :

$$0 \equiv \text{Max } \{E_t u[c, l, t] dt + E_t dV\} \quad (33)$$

This equation can be simplified because  $Z$  is a standard Wiener process. Thus  $E_t [dZ(t)] = 0$  and  $E_t [\sigma_k V_k + \sigma_W V_W + \sigma_Y V_Y] dZ(t) = 0$ . We can then write :

$$E_t dV = \left[ V_t + \mu_k V_k + \mu_W V_W + \mu_Y V_Y + \frac{1}{2} \sigma_k^2 V_{kk} + \frac{1}{2} \sigma_W^2 V_{WW} \right. \\ \left. + \frac{1}{2} \sigma_Y^2 V_{YY} + \sigma'_k \sigma_W V_{kW} + \sigma'_k \sigma_Y V_{kY} + \sigma'_Y \sigma_W V_{YW} \right] dt \quad (34)$$

By using this equation (34) and the equation

$$E_t u[c, l, t] = u[c, l, t] \quad (35)$$

one obtains an equation equivalent to (33), which is written :

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<sup>22</sup>That is the sum of terms in  $(dt)^\alpha$  with  $\alpha > 1$ .

$$\begin{aligned}
0 \equiv & \text{Max} \{u[c, l, t] dt \\
& + \left[ V_t + \mu_k V_k + \mu_W V_W + \mu_Y V_Y + \frac{1}{2} \sigma_k^2 V_{kk} + \frac{1}{2} \sigma_W^2 V_{WW} \right. \\
& \left. + \frac{1}{2} \sigma_Y^2 V_{YY} + \sigma'_k \sigma_W V_{kW} + \sigma'_k \sigma_Y V_{kY} + \sigma'_Y \sigma_W V_{YW} \right] dt \} \quad (36)
\end{aligned}$$

Parameters associated with Itô's process are as follows :

$$\begin{aligned}
\mu_k &= (\mu_h + \mu_\omega + (\mu_\theta + \sigma_{\theta\omega}) \lambda - \mu_\delta - \sigma_{\delta\omega}) \\
\mu_W &= rW - c + (1 - \lambda - l) k + W (\mu - r1)' X \\
\sigma_k^2 &= (\sigma_\omega + \sigma_\theta \lambda - \sigma_\delta)' (\sigma_\omega + \sigma_\theta \lambda - \sigma_\delta) k^2 \\
\text{since } \sigma_k &= k (\sigma_\omega + \sigma_\theta \lambda - \sigma_\delta)' \\
\sigma_W^2 &= W^2 X' \sigma'_s \sigma_s X \text{ since } \sigma_W = W X' \sigma'_s \\
\sigma'_k \sigma_W &= kW (\sigma_\omega + \sigma_\theta \lambda - \sigma_\delta)' \sigma_s X \\
\sigma'_k \sigma_Y &= k (\sigma_\omega + \sigma_\theta \lambda - \sigma_\delta)' \sigma_Y \\
\sigma'_Y \sigma_W &= W X \sigma'_Y \sigma_s
\end{aligned}$$

By dividing the two members of (36) by  $dt$  and replacing  $\mu_k$ ,  $\mu_W$ ,  $\sigma_k^2$ ,  $\sigma_W^2$ ,  $\sigma'_k \sigma_W$ ,  $\sigma'_k \sigma_Y$  and  $\sigma'_Y \sigma_W$ , by their above expression, one obtains the final version of the initial maximization program (12), which corresponds to equation (16) in the text :

$$\begin{aligned}
0 \equiv & \text{Max} \{u[c, l] + V_k (\mu_\omega + \mu_\theta \lambda + \lambda \sigma'_\theta \sigma_\omega - \delta - \sigma'_\omega \sigma_\delta) k \\
& + V_W [rW + (1 - \lambda - l) k - c + W X (\mu - r)] \\
& + \frac{1}{2} V_{WW} W^2 X^2 \sigma_s^2 + \frac{1}{2} V_{kk} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta) k^2 \\
& + V'_Y \mu_Y + \frac{1}{2} V_{YY} \sigma'_Y \sigma_Y + W X V_{WY} \sigma'_s \sigma_Y + V_{kW} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_s k W X \\
& + V_{kY} k (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_Y - \rho V + V_t \} \quad (37)
\end{aligned}$$

### A.3 Explicit solutions

$V(\cdot)$  designate the value function defined by:

$$J(t, W, k, u) = e^{-\rho t} V(t, W, k, u)$$

Using the same technique descibed in the previvious appendix, we know that programm (25) is equivalnent to the following Hamilton-Jacobi-Bellman

equation :

$$\begin{aligned}
0 \equiv & \text{Max} \{U[c, l] + V_k (\mu_\omega + \mu_\theta \lambda - \mu_\delta + \lambda \sigma'_\theta \sigma_\omega - \sigma'_\delta \sigma_\omega) k \\
& + V_W [rW + (1 - \lambda - l - \mu_u) k - c + WX (\mu - r) - kWX \sigma'_s \sigma_u] \\
& + \frac{1}{2} V_{WW} [W^2 X^2 \sigma'_s \sigma_s - 2kWX \sigma'_s \sigma_u + k^2 \sigma'_u \sigma_u] + \frac{1}{2} V_{kk} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta) k^2 \\
& + V'_u \mu_u + \frac{1}{2} V_{uu} \sigma'_u \sigma_u + V_{Wu} [WX \sigma'_s \sigma_u - k \sigma'_u \sigma_u] + V_{kW} (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' [WX \sigma'_s - k \sigma_u] k \\
& + V_{ku} k (\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_u - \rho V + V_t \}
\end{aligned} \tag{38}$$

The first ordinary conditions guive the four solutions:

$$U_c - V_W = 0 \tag{39}$$

$$U_l - V_W \mu_u k = 0 \tag{40}$$

$$X = -\frac{V_W}{V_{WW}W} \frac{(\mu - r) - \sigma'_s \sigma_u}{\sigma'_s \sigma_s} + \frac{k}{W} \frac{\sigma'_s \sigma_u}{\sigma'_s \sigma_s} - \frac{V_{Wu}}{V_{WW}W} \frac{\sigma'_s \sigma_u}{\sigma'_s \sigma_s} - \frac{V_{kW}}{V_{WW}W} \frac{(\sigma_\omega + \lambda \sigma_\theta - \sigma_\delta)' \sigma_s}{\sigma'_s \sigma_s} k \tag{41}$$

$$\lambda = \frac{-V_k}{V_{kk}k} \frac{1}{\sigma'_\theta \sigma_\theta} \left( \mu_\theta + \sigma'_\theta \sigma_\omega - \frac{V_W}{V_k} \right) - \frac{(\sigma'_\omega \sigma_\theta - \sigma'_\delta \sigma_\theta)}{\sigma'_\theta \sigma_\theta} - \frac{V_k}{V_{kk}k} \left[ \frac{V_{ku} \sigma'_\theta \sigma_u}{V_k \sigma'_\theta \sigma_\theta} - \frac{V_{kW} \sigma'_\theta \sigma_u k}{V_k \sigma'_\theta \sigma_\theta} \right] \tag{42}$$

If preferences are described by the instantaneous log utility function:

$$U = \phi_c \ln c + (1 - \phi_c) \ln l$$

the general solution of progrmm (25) is the value function :

$$V(k, t, W, uT) = A(t, T) \ln [B(t, T) k + W - C(t, T) u] \tag{43}$$

$$\begin{aligned}
V_W &= A(t, T) [B(t, T) k + W - C(t, T) u]^{-1} \\
V_k &= A(t, T) B(t, T) [B(t, T) k + W - C(t, T) u]^{-1} \\
V_u &= -A(t, T) C(t, T) [B(t, T) k + W - C(t, T) u]^{-1} \\
V_{WW} &= -A(t, T) [B(t, T) k + W - C(t, T) u]^{-2} \\
V_{kk} &= -A(t, T) B(t, T)^2 [B(t, T) k + W - C(t, T) u]^{-2} \\
V_{uu} &= -A(t, T) C(t, T)^2 [B(t, T) k + W - C(t, T) u]^{-2} \\
V_{Wk} &= -A(t, T) B(t, T) [B(t, T) k + W - C(t, T) u]^{-2} \\
V_{Wu} &= A(t, T) C(t, T) [B(t, T) k + W - C(t, T) u]^{-2} \\
V_{ku} &= A(t, T) B(t, T) C(t, T) [B(t, T) k + W - C(t, T) u]^{-2} \\
\\ 
-\frac{V_{kk}k}{V_k} &= \frac{B(t, T) k}{[B(t, T) k + W - C(t, T) u]} < 1 \\
-\frac{V_{WW}W}{V_W} &= \frac{W}{[B(t, T) k + W - C(t, T) u]} < 1
\end{aligned}$$

Replacing these results into general solutions (39), (40), (41), (42), we obtain explicit solution of programm (25) :

$$c = \frac{\phi_c}{A(t, T)} [B(t, T) k + W - C(t, T) u] \quad (44)$$

$$l = \frac{1 - \phi_c}{A(t, T) \mu_u k} [B(t, T) k + W - C(t, T) u] \quad (45)$$

$$X = \frac{[B(t, T) k + W - C(t, T) u]}{W} \frac{(\mu - r) - \sigma'_s \sigma_u}{\sigma'_s \sigma_s} + \frac{k}{W} \frac{\sigma'_s \sigma_u}{\sigma'_s \sigma_s} + \frac{C(t, T)}{W} \frac{\sigma'_s \sigma_u}{\sigma'_s \sigma_s} - \frac{B(t, T)}{W} \frac{(\sigma_\omega - \sigma_\delta)' \sigma_s}{\sigma'_s \sigma_s} k \quad (46)$$

$$\lambda = \frac{[B(t, T) k + W - C(t, T) u]}{B(t, T) k} \frac{1}{\sigma'_\theta \sigma_\theta} \left( \mu_\theta + \sigma'_\theta \sigma_\omega - \frac{1}{B(t, T)} \right) - \frac{(\sigma'_\omega \sigma_\theta - \sigma'_\delta \sigma_\theta)}{\sigma'_\theta \sigma_\theta} + \frac{1 + C(t, T)}{B(t, T)} \frac{\sigma'_\theta \sigma_u}{\sigma'_\theta \sigma_\theta} \quad (47)$$